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Engineering quantum cloning through maximal entanglement between boundary qubits in an open spin chain

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Abstract
In this paper, we show how maximal entanglement between boundary qubits in the open spin chain of an *XX* model is realized. This creation of maximal entanglement could be used for phase covariant quantum cloning in a spin chain. The maximal entanglement is achieved with specially engineered couplings. We compare our realization with alternative methods and find that the method of pre-engineered couplings is straightforward and the decrease of cloning fidelity due to time errors is smaller.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum teleportation [1] and quantum cloning [2] are two significant breakthroughs in quantum information theory over the last 10 years. In quantum teleportation, one can transfer an unknown state from one location to another provided that the two locations share an entangled resource. Quantum cloning however is restricted by the no-cloning theorem. The no-cloning theorem [3] states that there is no quantum operations that can duplicate an arbitrary quantum state perfectly. However, it is still possible to have imperfect cloning [4]: in imperfect cloning, one simply duplicates states of high fidelity. In particular, for a symmetric universal N copy to M copy quantum cloning machine [5], it can be shown that the optimal fidelity is $F_{N\to M} = \frac{MN+M+N}{M(N+2)}$. For a $1 \to 2$ quantum cloning machine, the optimal fidelity is $\frac{5}{6}$ [6–8].

The optimal fidelity of $5/6$ is not sacrosanct: one can achieve far better optimal fidelity by restricting the qubits to certain states, for instance the states that lie on the equator. For symmetric phase-covariant cloning [8, 9], a minimum fidelity of $F \approx 0.8536$ can be achieved. Like quantum teleportation, quantum cloning can be realized experimentally in the laboratory through quantum gates and circuits. Indeed, the optical implementation of the $1 \to 2$ cloners, based on parametric downconversion as the amplification phenomenon, was successfully demonstrated [10, 11].

More recently, increasing attention has been devoted to a spin chain as a fundamental communication structure [12]. It has been shown that it is possible to transfer an arbitrary state without a shared resource through a spin chain. Such perfect state transfer (PST) in a spin chain can be realized through a pre-engineered couplings [13, 14]. The important feature of this approach is that once the coupling constants are fixed to certain values, the state of the qubit at one end will be transferred to the other end perfectly at certain time regardless of the length of the chain.

In this paper, we look at the feasibility of cloning qubits through a spin chain. In short, we would like to find out whether or not it is possible to share the information equally between the end (boundary) qubits, starting with an arbitrary state in the first qubit at the beginning of the process. Our question is essentially a resource-saving quantum cloning in a spin chain, which was first mentioned by Chen *et al* [16]. In their paper, they considered $1 \to M$ phase covariant cloning (PCC) [8, 15, 17] in a spin star network based on the *XXZ* model and found that resource-saving PCC can be achieved if M is an even number. For the PCC, the required unitary evolution is as follows[5]:

$$U|00\rangle = |00\rangle, \tag{1}$$

$$U|10\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle). \tag{2}$$

Suppose that the input state for PCC is $|\psi\rangle = |\varphi_1\rangle|0\rangle$, where $|\varphi_1\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi}|1\rangle$ is the state to be copied. Then the fidelity for PCC is $f(\theta) = \langle\varphi_1|\rho_{\mathrm{out}}|\varphi_1\rangle$, where $\rho_{\mathrm{out}} = Tr_2[U|\psi\rangle\langle\psi|U^{\dagger}]$ is one of the symmetric output states. The fidelity can be calculated simply as $f(\theta) = (5 + \sqrt{2} + 2\cos\theta - (\sqrt{2} - 1)\cos 2\theta)/8$, which is independent of phase ϕ. The dependence of fidelity on θ is shown in figure 1, and it can be seen that the fidelity reaches minimum value $(2 + \sqrt{2})/4 \approx 0.8536$ when $\theta = \pi/2$, i.e. $|\varphi_1\rangle = (|0\rangle + \mathrm{e}^{\mathrm{i}\phi}|1\rangle)/\sqrt{2}$ which lies on the equator of the Bloch sphere. PCC has important applications in quantum cryptography, for instance, in BB84 protocol [18], where it provides the optimal eavesdropping strategy for Eve to acquire information of the qubit sent by Alice [19].

We note that this cloning process is essentially the generation of maximal entanglement between the boundary qubits[6]. Entanglement generation through spin interactions is also important for realizing solid-state quantum computation [21–23]. Equation (1) is trivial and is automatically satisfied in the spin chain of an *XX* model, since the state $|0\rangle^{\otimes N}$ is an eigenstate of the Hamiltonian (cf (3) below) and thus will not evolve with time. Equation (2) is the crucial one, which means that flipping one of the two boundary qubits will generate maximal entanglement between them. Therefore, realizing PCC amounts to the generation of entanglement between the two boundary qubits when one of the qubits is flipped. Can this property be achieved in a linear spin chain? We analyze this question carefully in the following

[5] $U|01\rangle$ and $U|11\rangle$ can be defined similarly, which are omitted. Here, we consider the state-dependent quantum cloning with partial information that the Bloch vector of the state is in the northern hemisphere, cf [17].

[6] The spin chain realizing perfect state transfer can also realize maximal entanglement generation of end qubits with appropriate initial product states but in that case resource-saving cloning cannot be fulfilled. See [20, 26] for details.

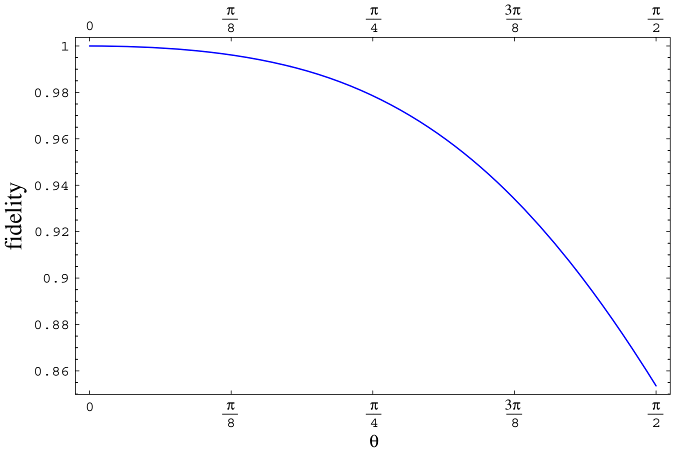

Figure 1. The fidelity of state-dependent phase covariant quantum cloning as a function of the input state angle θ. The input state is $|\psi\rangle = |\varphi_1\rangle|0\rangle$, where $|\varphi_1\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ is the state to be copied.

sections. The paper is organized as follows. In section 2, we discuss the *XX* model with pre-engineered couplings and give a general formula for the coupling constants. In section 3, we compare our cloning method with an alternative one based on the clone-and-swap scheme. In section 4, we give a proof for our general result. In section 5, we give a conclusion to finish this paper.

2. *XX* model with pre-engineered couplings

The Hamiltonian of our model is

$$H = \sum_{i=1}^{N-1} \frac{J_i}{2}\left[\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y\right] + \sum_{i=1}^{N} B_i\sigma_i^z. \tag{3}$$

Without loss of generality, we assume that all J_i's are real and positive[7]. This model can be experimentally realized through manipulations of control lasers and detuning in coupled atom-cavity arrays [24] or through controlling external voltage in linear arrays of tunnel-coupled quantum dots [25]. Since the operator of the total z component of the spin: $\sigma_{\rm tol}^z = \sum_{i=1}^N \sigma_i^z$ commutes with the Hamiltonian, the state $|100\ldots0\rangle$ must evolve to a superposition of states with only one up-spin. Therefore, we can work in the one excitation subspace spanned by the basis vectors $|n\rangle, n = 1, 2, \ldots, N$, where $|n\rangle$ denotes the state with an up-spin at nth qubit [13]. The Hamiltonian in this subspace has the following matrix form:

$$H = \begin{pmatrix} -2B_1 & J_1 & 0 & \ldots & 0 \\ J_1 & -2B_2 & J_2 & \ldots & 0 \\ 0 & J_2 & -2B_3 & \ldots & \vdots \\ \vdots & \vdots & \vdots & \ddots & J_{N-1} \\ 0 & 0 & 0 & J_{N-1} & -2B_N \end{pmatrix} + \left(\sum_{i=1}^N B_i\right) I_{N\times N}, \tag{4}$$

[7] It can be proved that for any open spin chain, there always exists a local unitary transformation for the bases such that the phases of J_i's are absorbed into the bases.

where $I_{N\times N}$ is an $N \times N$ identity matrix. For the entanglement of boundary qubits, we would like to realize the following time evolution:

$$U = \mathrm{e}^{-\mathrm{i}tH}|1\rangle = \frac{\mathrm{e}^{\mathrm{i}\phi_1}|1\rangle + \mathrm{e}^{\mathrm{i}\phi_2}|N\rangle}{\sqrt{2}}, \tag{5}$$

where ϕ_1 and ϕ_2 are some arbitrary phases[8]. By redefining $H' = Ht$, one can set t to unity. For constant J_i, it can be verified that equation (5) is satisfied only for $N = 2$ or 3. The relevant results are presented as follows: (i) for $N = 2$, $B_1 = B_2$, $J_1 = \frac{\pi}{4} + \frac{k}{2}\pi$, where k is a non-negative integer; (ii) for $N = 3$, $B_1 = B_3 = \frac{\pi}{2}$, $B_2 = 0$, $J_1 = J_2 = \frac{\sqrt{6}\pi}{4}$. It is interesting to note that in both cases, $|\langle N|U^2|1\rangle| = 1$, so that at time $t = 2$, perfect state transfer occurs.

For pre-engineered couplings (i.e. different J_i), we can obtain solutions satisfying equation (5). In particular for $B_i = 0$ for all i, an interesting solution for arbitrary length of spin chain N is[9]

$$J_{N,i} = f_{N,i}\frac{\sqrt{i(N-i)}}{2}\pi, \tag{6}$$

where

$$f_{N,i} = \begin{cases} \sqrt{1 \pm \left(\delta_{i,\frac{N-1}{2}} - \delta_{i,\frac{N+1}{2}}\right)\frac{1}{\sqrt{2}}}, & \text{if } N \text{ is odd,} \\ \sqrt{\dfrac{\left(N-2i+1\pm\frac{1}{2}\right)\left(N-2i-1\mp\frac{1}{2}\right)}{(N-2i+1)(N-2i-1)}}, & \text{if } N \text{ is even.} \end{cases} \tag{7}$$

In equation (7), $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise. Note that we have two solutions in equation (7). The first solution corresponds to the upper signs in $\pm$ and $\mp$ in equation (7), and the second solution corresponds to the lower signs. It can be seen that the coupling constants depend on both N and i, which are equal to those of the PST case $\left(J_{N,i} = \frac{\sqrt{i(N-i)}}{2}\pi\right)$ multiplied by a factor $f_{N,i}$. This factor has some interesting properties. For the odd N case, $f_{N,\frac{N-1}{2}} \neq f_{N,\frac{N+1}{2}}$, and thus $J_{N,\frac{N-1}{2}} \neq J_{N,\frac{N+1}{2}}$, which means that the central two couplings are asymmetric with respect to the central qubit. For the even N case, if we consider the coupling in the middle of the chain, i.e. $i = \frac{N}{2}$, we see that $f_{N,i} = 1 \pm \frac{1}{2} = \frac{1}{2}, \frac{3}{2}$. For $i = \frac{N}{2} - m$, with m being an integer enumerated from the middle of the chain, $f_{N,i} = \sqrt{\frac{(2m+1\pm\frac{1}{2})(2m-1\mp\frac{1}{2})}{(2m+1)(2m-1)}}$. These factors asymptotically go to unity as $m \to \infty$. Figure 2 shows the values of the factors $f_{N,i}$ as functions of m. Thus, for large N, the couplings near the boundary qubits approach the values for PST.

We illustrate our results with $N = 60, 61$. Figure 3 shows the magnitude of the couplings for $N = 60$ and $N = 61$. It can be seen that all the four plots contain jumps at the center. Figures 4 and 5 show the time dependence of the probability amplitudes of certain qubits using one of the two possible solutions for $N = 60$ and $N = 61$, respectively. It can be seen that at $t = 1$ in both figures, the amplitudes for boundary qubits are 0.707, meaning that they are maximally entangled. At $t = 2$ in $N = 60$ case, the amplitude of the 60th qubit is 1, which indicates that PST occurs from the first qubit to the last one. However, there is no PST in the odd N case. This phenomenon is closely related to the asymmetric couplings required in the odd case. Indeed, perfect state transfer in a spin chain requires symmetric couplings [26].

[8] ϕ_1 and ϕ_2 do not affect the cloning fidelity, since they can be absorbed into the bases through a redefinition, e.g. $|1'\rangle = \mathrm{e}^{\mathrm{i}\phi_1}|1\rangle$, etc.

[9] The solution is not unique. Nevertheless, our solution deserves special attention as it is closely related to the solution of PST, e.g. $J_i = \frac{\sqrt{i(N-i)}}{2}\pi$. Actually, this closed relation always exists for any solutions of PST in a spin chain of an odd number of qubits. The proof will be presented in a later publication which is in preparation.

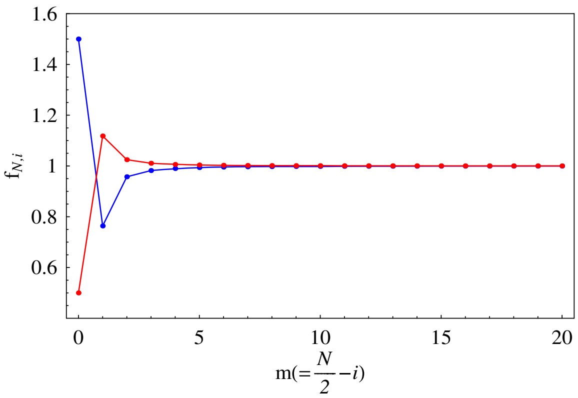

Figure 2. The factor $f_{N,i}$ as functions of $m = \frac{N}{2} - i$. The red and blue lines correspond to two solutions of $f_{N,i}$ of the even N case.

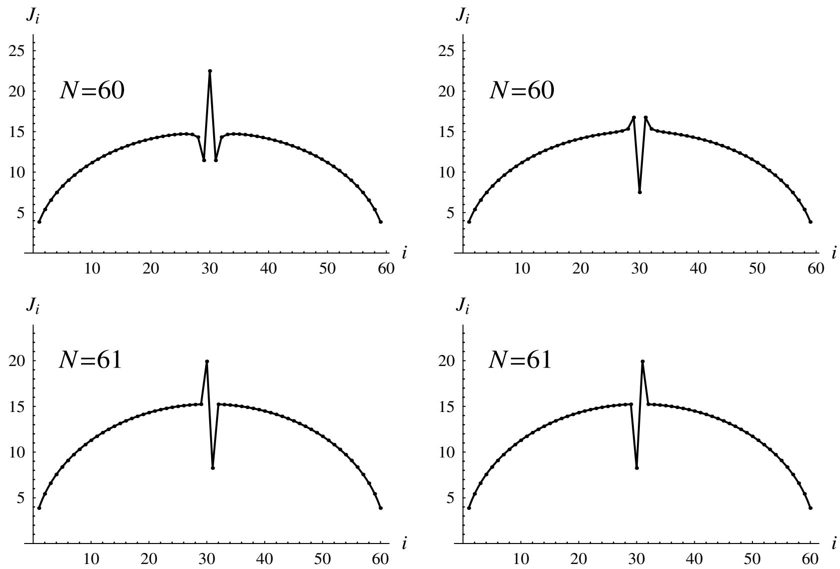

Figure 3. Patterns of J_i's. The upper two figures are the first and second solutions of $N = 60$ case respectively, and the lower two figures are the first and second solutions of the $N = 61$ case, respectively. It can be seen that there is a reflection symmetry of J_i's in the even N case, while the symmetry of J_i's is breaking if N is odd.

In the case of an odd spin chain, for any given couplings that permit PST, the introduction of asymmetry in the middle two couplings always results in maximal entanglement between the boundary qubits (also cf footnote 9). This introduction of asymmetric couplings is analogous to an insertion of a beam splitter[10]. To see this connection, note that the ratio between the

[10] The concept of a beam splitter in a spin chain has already been proposed in [27, 28] for a two-dimensional case. Here we briefly discuss this concept in a one-dimensional spin chain.

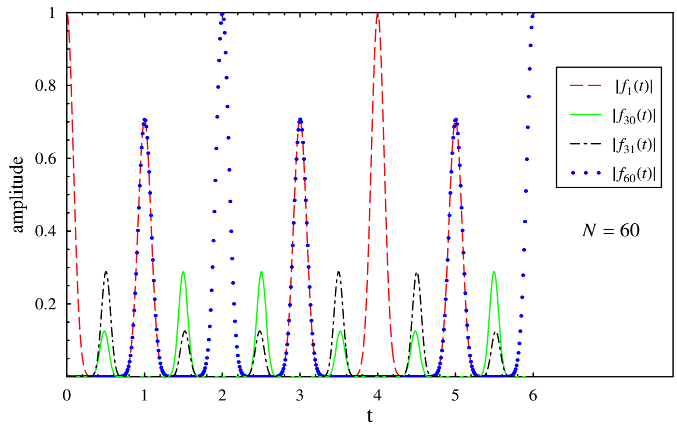

Figure 4. Time dependence of amplitudes of four certain qubits in a 60-qubit spin chain of the *XX* model using the first solution of the $N = 60$ case. $H = \sum_{i=1}^{59} \frac{J_i}{2}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y)$. The state is $|\phi(t)\rangle = \sum_{i=1}^{60} f_i(t)|i\rangle$, and $|\phi(0)\rangle = |1\rangle$. The red, green, black and blue lines are for $|f_i(t)|$ $(i = 1, 30, 31, 60)$, respectively.

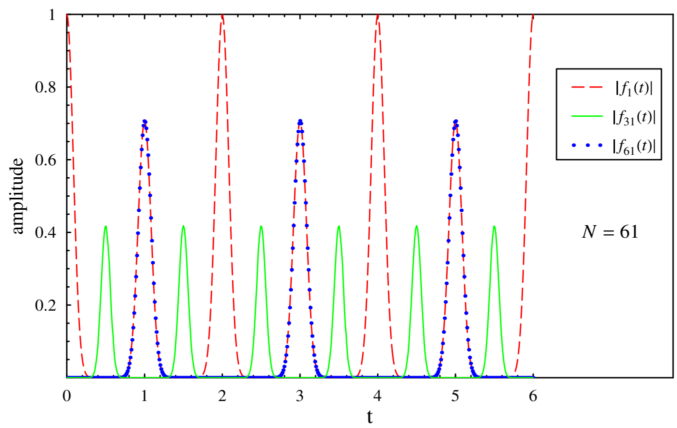

Figure 5. Time-dependence of amplitudes of three certain qubits in a 61-qubit spin chain of the *XX* model using the first solution of the $N = 61$ case. $H = \sum_{i=1}^{60} \frac{J_i}{2}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y)$. The state is $|\phi(t)\rangle = \sum_{i=1}^{61} f_i(t)|i\rangle$, and $|\phi(0)\rangle = |1\rangle$. The red, green and blue lines are for $|f_i(t)|$, $(i = 1, 31, 61)$, respectively.

central two couplings is $\sqrt{2} \pm 1$, as shown in equations (6) and (7). This ratio is independent of the number of qubits in a spin chain and can be obtained simply by considering a 3-qubit spin chain with the Hamiltonian given by

$$H = \begin{pmatrix} 0 & J_1 & 0 \\ J_1 & 0 & J_2 \\ 0 & J_2 & 0 \end{pmatrix}$$

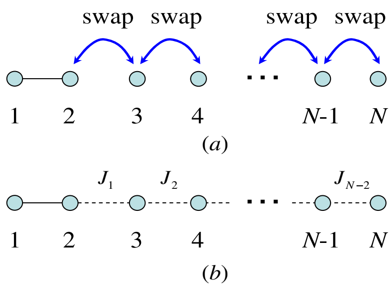

Figure 6. Quantum cloning based on (a) swap operations and (b) pre-engineered couplings where $J_i = \sqrt{i(N-1-i)}$ is for PST in [13].

and demanding that $e^{-iH} = U$ be a Hadamard matrix (beam splitter) in the one-excitation subspace of the first and third qubit, i.e.[11]

$$U \sim \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Moreover, it can be verified that $e^{-iH}|i\rangle = \frac{e^{i\phi_1}|i\rangle + e^{i\phi_2}|N+1-i\rangle}{\sqrt{2}}$ $\left(i = 1, 2, \ldots, \frac{N-1}{2}\right)$. This result indicates that the central asymmetry can always split a spin excitation wherever this excitation is (at one side of the central asymmetry), justifying the analogy of the central asymmetry to a beam splitter.

3. Comparison with an alternative method

An alternative method for quantum cloning of boundary qubits is as follows: first, we perform the cloning in a 2-qubit spin chain. Second, we transfer the quantum state of the second qubit in the above spin chain to the spin at the right end. The second step can be realized either through successive swap operations (figure 6(*a*), referred to as clone-and-swap scheme) or through pre-engineered coupling constants for PST (figure 6(*b*)) after the state has been cloned using the first two qubits.

Compared with the clone-and-swap scheme, our method is straightforward as one only needs to prepare the initial state of a spin chain, while in the clone-and-swap scheme, swap operations must be performed sequentially one after another in order to transfer the state of qubit 2 to qubit N. Moreover, the errors may accumulate during these successive operations, which can be seen as follows. These errors usually originate from (i) interactions between qubits in a spin chain with environment; (ii) imprecise time determination of operations. Error (i) occurs in all schemes and has been discussed extensively elsewhere [26, 29]. For simplicity and illustrative purpose, we only consider error (ii). Such errors could result from inherent time resolution of experimental apparatus or set-up.

Let us analyze the swap operations of the clone-and-swap scheme in some detail. In this scheme, the couplings need to switch on and off many times. Suppose the ideal waveform for the switch is a rectangular pulse. In practice, however, this pulse is distorted, as seen in figure 7. This distortion could render the determination of the operation time imprecise.

[11] The actual matrix form of U may differ from the form in the text in some phases before each matrix element as long as the matrix is unitary and the modulus of each matrix element is the same as the one in the text.

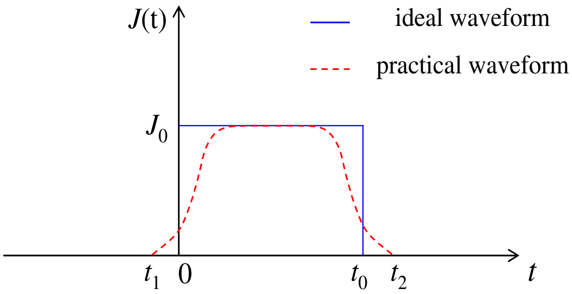

Figure 7. The ideal and practical waveforms of one of the couplings in the spin chain using the clone-and-swap scheme for illustrative purpose.

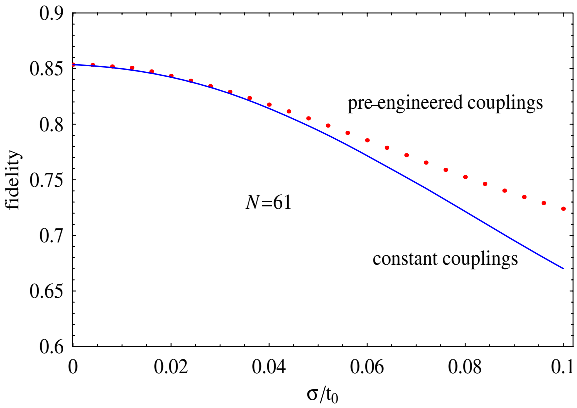

Figure 8. The cloning fidelity as a function of the parameter σ/t_0 for the two schemes with $N = 61$. The input state is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

Assume that the ideal switch-on time is t_0, and the practical time is $t_2 - t_1$. The swap operation is performed on the 2-qubit chain with the Hamiltonian $H_2 = J(t)|+\rangle\langle+| - J(t)|-\rangle\langle-|$, where $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$. The relevant dynamics is

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\int_{t_1}^{t_2} H_2\,\mathrm{d}t} &= \mathrm{e}^{-\mathrm{i}\int_{t_1}^{t_2} J(t)\,\mathrm{d}t}|+\rangle\langle+| + \mathrm{e}^{\mathrm{i}\int_{t_1}^{t_2} J(t)\mathrm{d}t}|-\rangle\langle-| \\ &= \mathrm{e}^{-\mathrm{i}J_0 t_0(1+\delta t_0)}|+\rangle\langle+| + \mathrm{e}^{\mathrm{i}J_0 t_0(1+\delta t_0)}|-\rangle\langle-| = \mathrm{e}^{-\mathrm{i}H_{20}t_0(1+\delta t_0)}, \end{aligned} \tag{8}$$

where $H_{20} = J_0|+\rangle\langle+| - J_0|-\rangle\langle-|$, J_0 is the ideal constant coupling and t_0 is the corresponding time for the swap operation.

From equation (8), it can be seen that an imperfect pulse in the coupling gives rise to an error δt_0 in the required operation time. For simplicity, suppose for both of the two schemes the time error obeys the Gaussian distribution $N(0, \sigma/t_0)$. Figure 8 shows the fidelity as a function of errors, i.e. the standard deviation of operating time σ/t_0 for the two schemes in an $N = 61$ spin chain. For our simulation, we assume that the input state is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$. It can be seen that the fidelity for pre-engineered couplings is better than the clone-and-swap scheme, especially for larger errors. Figure 9 shows the fidelity as a function of the length of the chain with a fixed error $\sigma/t_0 = 0.1$ for the two schemes, again with the input state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$. It can

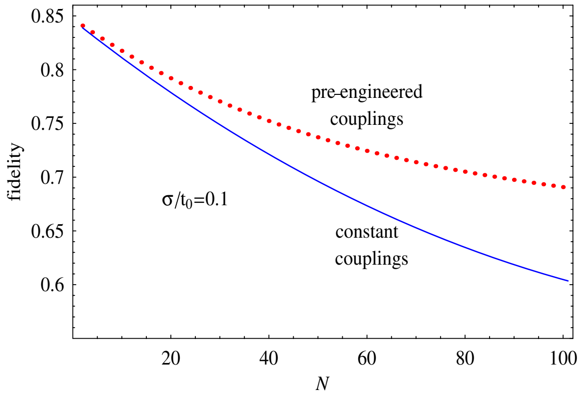

Figure 9. The cloning fidelity as a function of the length of a spin chain for our scheme of pre-engineered couplings and the clone-and-swap scheme of constant couplings, respectively. The input state is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

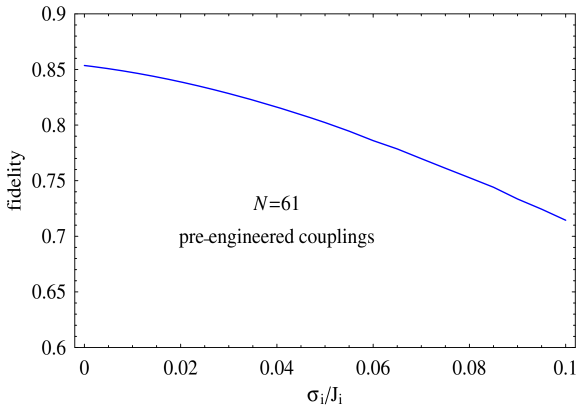

Figure 10. The cloning fidelity as a function of the the couplings' error parametrized by $\sigma_i/J_i = \delta_0$ for the scheme of pre-engineered couplings in an $N = 61$ spin chain. The input state is $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

be seen that for a long spin chain, pre-engineered couplings again perform better than the clone-and-swap scheme.

It is important to note that in the scheme of pre-engineered couplings, imprecision in values for couplings may also result in a decrease in cloning fidelity. This is shown in figure 10, where we have assumed that J_i obeys the Gaussian distribution $N(J_i, \sigma_i)$ and $\sigma_i/J_i = \delta_0$, i.e. the same for all i. It can be seen that when $\sigma_i/J_i = 0.1$, i.e. 10% error, the fidelity decreases to 0.714. This decrease in fidelity is still tolerable, showing that pre-engineered couplings could be quite robust against the errors even for practical applications.

Figure 6(*b*) is essentially the same as the scheme for cloning with pre-engineered couplings. However, the latter scheme still has a slight advantage: instead of performing two steps to realize cloning, in the latter method we require only one step.

4. Proof of our general result

In this section, we provide a proof of our general result. As mentioned in section 2, $\left[\sigma_{\mathrm{tol}}^{z}, H\right] = 0$. So we can work in the subspace spanned by the basis vectors $|n\rangle$, $n = 1, 2, \ldots, N$. In this subspace, for the first solution of the odd N case, construct the following unitary operator:

$$U_0|j\rangle = (-1)^{\frac{N-1}{2}} \frac{(-1)^f|j\rangle + |N+1-j\rangle}{\sqrt{2}}, \qquad j \neq \frac{N+1}{2}, \tag{9}$$

$$U_0\left|\frac{N+1}{2}\right\rangle = (-1)^{\frac{N-1}{2}}\left|\frac{N+1}{2}\right\rangle, \tag{10}$$

where $f = \theta[j-(N+1)/2]$ is a step function ($f = 0$ for $j \leqslant (N+1)/2$, $f = 1$ otherwise). The eigenvalues of U_0 are ± 1. The number of -1's is $2\left\lfloor\frac{N+1}{4}\right\rfloor$, where $\left\lfloor\frac{N+1}{4}\right\rfloor$ denotes the maximum integer less than or equal to $(N+1)/4$. The eigenvectors of U_0 are

$$|\lambda_j\rangle = \begin{cases} \dfrac{\left(1+\sqrt{2}\lambda_j(-1)^{\frac{N-1}{2}}\right)|j\rangle + |N+1-j\rangle}{\sqrt{4+2\sqrt{2}\lambda_j(-1)^{\frac{N-1}{2}}}} & \text{if} \quad 1 \leqslant j \leqslant \frac{N-1}{2}, \\ \left|\dfrac{N+1}{2}\right\rangle & \text{if} \quad j = \dfrac{N+1}{2}, \end{cases} \tag{11}$$

where λ_j are the corresponding eigenvalues and j denotes the jth qubit in the spin chain. The corresponding Hamiltonian to U_0 is (setting time t equal to unity)

$$H_0 = i \cdot \log[U_0] = \sum_{i=1}^{2\lfloor\frac{N+1}{4}\rfloor} (2k_i+1)\pi\Gamma_i^- + \sum_{j=1}^{N-2\lfloor\frac{N+1}{4}\rfloor} 2k_j\pi\Gamma_j^+, \tag{12}$$

where k_i's are integers $\Gamma_i^- = |\theta_{-1i}\rangle\langle\theta_{-1i}|$, $\Gamma_j^+ = |\theta_{+1j}\rangle\langle\theta_{+1j}|$ and

$$|\theta_{-1i}\rangle = \sum_{j=1}^{2\lfloor\frac{N+1}{4}\rfloor} R_{1i}^{j}|\lambda_j = -1\rangle, \tag{13}$$

$$|\theta_{+1j}\rangle = \sum_{k=1}^{N-2\lfloor\frac{N+1}{4}\rfloor} R_{2j}^{k}|\lambda_k = +1\rangle. \tag{14}$$

The matrices R_{1i}^{j} and R_{2j}^{k} are unitary transformations in the respective degenerate subspace. It is interesting to note that R_{1i}^{j} is an element of the group $SU\left(2\left\lfloor\frac{N+1}{4}\right\rfloor\right)$ and R_{2j}^{k} is an element of the group $SU\left(N-2\left\lfloor\frac{N+1}{4}\right\rfloor\right)$. The matrix form of equation (12) in the basis vectors of equation (11) is

$$H_0 = \left(\begin{array}{c|c} \boldsymbol{R_1 D_1 R_1^\dagger} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{R_2 D_2 R_2^\dagger} \end{array}\right), \tag{15}$$

where $\boldsymbol{D_1} = \mathrm{Diag}[(2k_1+1)\pi, (2k_2+1)\pi, \ldots]$, $\boldsymbol{D_2} = \mathrm{Diag}[2k_1\pi, 2k_2\pi, \ldots]$.

The Hamiltonian (3) in the basis vectors of equation (11) has the following matrix form[12]:

$$H' = \left(\begin{array}{c|c} \boldsymbol{X_{m+1}} & \boldsymbol{Y} \\ \hline \boldsymbol{Y^T} & \boldsymbol{Z_m} \end{array}\right), \tag{16}$$

[12] Here we do not present the explicit form of the matrix elements of H'. But it can be observed that the matrix $\boldsymbol{Y}$ contains terms: $\frac{-J_i+J_{N-i}}{2\sqrt{2}}$, for $1 \leqslant i \leqslant \frac{N-3}{2}$, and $\frac{\sqrt{2+\sqrt{2}}}{2}[(1-\sqrt{2})J_i + J_{i+1}]$, for $i = \frac{N-1}{2}$.

where $m = \frac{N}{2}$ and m is odd. This matrix must be equal to the matrix (15) with some $\boldsymbol{R_1}$, $\boldsymbol{R_1}$, $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$. Therefore, $\boldsymbol{Y}$ in (16) must be $\mathbf{0}$. From this, we get

$$J_i = J_{N-i}, \qquad \left(1 \leqslant i \leqslant \frac{N-3}{2}\right), \tag{17}$$

$$J_i = (\sqrt{2}+1)J_{i+1}, \qquad \left(i = \frac{N-1}{2}\right). \tag{18}$$

Using (17) and (18) to simplify (16), we obtain

$$H' = \left(\begin{array}{c|c} \boldsymbol{X'_{m+1}} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{Z'_m} \end{array}\right), \tag{19}$$

where

$$\boldsymbol{X'_{m+1}} = \begin{pmatrix} 0 & J_1 & 0 & \cdots & 0 \\ J_1 & 0 & J_2 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \sqrt{4-2\sqrt{2}}J_m \\ 0 & 0 & 0 & \sqrt{4-2\sqrt{2}}J_m & 0 \end{pmatrix} \tag{20}$$

$$\boldsymbol{Z'_m} = \begin{pmatrix} 0 & J_1 & 0 & \cdots & 0 \\ J_1 & 0 & J_2 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{m-1} \\ 0 & 0 & 0 & J_{m-1} & 0 \end{pmatrix}. \tag{21}$$

For even m, exchange $\boldsymbol{X'_{m+1}}$ and $\boldsymbol{Z'_m}$ of (19).

In order to get the solution for the couplings, one needs to find $\boldsymbol{R_1}$, $\boldsymbol{R_1}$, $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ in equation (15), which are not unique. We conjecture that $\boldsymbol{D_1} = \mathrm{diag}[m\pi, -m\pi, (m-2)\pi, -(m-2)\pi, \ldots, \pi, -\pi]$, $\boldsymbol{D_2} = \mathrm{diag}[(m-1)\pi, -(m-1)\pi, (m-3)\pi, -(m-3)\pi, \ldots, 0]$. If m is even, replace m with $m-1$ in $\boldsymbol{D_1}$, and replace m with $m+1$ in $\boldsymbol{D_2}$. Actually, the diagonal terms of $\boldsymbol{D_1}$ and $\boldsymbol{D_1}$ are the eigenvalues of the Hamiltonian (3). These eigenvalues are the same as those of the Hamiltonian for perfect state transfer (H_{pst}) with $J_i = \frac{\sqrt{i(N-i)}}{2}\pi$. This property, as can be seen later, helps to solve our problem. Rewriting H_{pst} in the representation of the eigenvectors of $\mathrm{e}^{-\mathrm{i}H_{\mathrm{pst}}}$, we get, for $N = 2m+1$, if m is odd,

$$H_{\mathrm{pst}} = \left(\begin{array}{c|c} \boldsymbol{K_{m+1}} & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{L_m} \end{array}\right), \tag{22}$$

where

$$\boldsymbol{K_{m+1}} = \begin{pmatrix} 0 & J_1^{\mathrm{pst}} & 0 & \cdots & 0 \\ J_1^{\mathrm{pst}} & 0 & J_2^{\mathrm{pst}} & \cdots & 0 \\ 0 & J_2^{\mathrm{pst}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \sqrt{2}J_m^{\mathrm{pst}} \\ 0 & 0 & 0 & \sqrt{2}J_m^{\mathrm{pst}} & 0 \end{pmatrix} \tag{23}$$

$$\boldsymbol{L}_m = \begin{pmatrix} 0 & J_1^{\mathrm{pst}} & 0 & \dots & 0 \\ J_1^{\mathrm{pst}} & 0 & J_2^{\mathrm{pst}} & \dots & 0 \\ 0 & J_2^{\mathrm{pst}} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{m-1}^{\mathrm{pst}} \\ 0 & 0 & 0 & J_{m-1}^{\mathrm{pst}} & 0 \end{pmatrix}. \tag{24}$$

If m is even, exchange $\boldsymbol{K}_{m+1}$ and $\boldsymbol{L}_m$ in (22). The general solution to $U_2 = \mathrm{e}^{-\mathrm{i}H_{\mathrm{pst}}}$ is (H_{pst} is only a special solution to U_2)

$$H_2 = \left(\begin{array}{c|c} \boldsymbol{R}_1'\boldsymbol{D}_1'\boldsymbol{R}_1'^{\dagger} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{R}_2'\boldsymbol{D}_2'\boldsymbol{R}_2'^{\dagger} \end{array}\right), \tag{25}$$

where the parameters are similar to those in equation (15). Since we have conjectured that the eigenvalues of (3) are the same as those of H_{pst} (i.e. $\boldsymbol{D}_1 = \boldsymbol{D}_1'$ and $\boldsymbol{D}_2 = \boldsymbol{D}_2'$) and the structure of (22) is the same as that of (19), we conclude that $\boldsymbol{R}_1 = \boldsymbol{R}_1'$ and $\boldsymbol{R}_2 = \boldsymbol{R}_2'$. Therefore, the matrix (22) is equal to the matrix (19). Thus,

$$J_i = J_i^{\mathrm{pst}}, \qquad (1 \leqslant i \leqslant m-1) \tag{26}$$

$$\sqrt{4 - 2\sqrt{2}}J_m = \sqrt{2}J_m^{\mathrm{pst}}. \tag{27}$$

This ends the proof of first solution for odd N.

The second solution for odd N can be obtained similarly. The only difference is that U_0 slightly changes and J_i exchanges with J_{i+1} for $i = (N-1)/2$. For the first solution of even N, U_0 in (9) and (10) does not work, which suggests we need to add some relative phase between antipodal qubits (i.e. qubit 1 and qubit N, qubit 2 and qubit $N-1$, etc):

$$U_0|j\rangle = \frac{|j\rangle + i(-1)^{\frac{N}{2}}|N+1-j\rangle}{\sqrt{2}}. \tag{28}$$

Using the method in the proof for odd N, the counterpart of (15) is ($N = 2k$)

$$H_{2k} = \left(\begin{array}{c|c} \boldsymbol{R}_1\boldsymbol{P}_k\boldsymbol{R}_1^{\dagger} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{R}_2\boldsymbol{Q}_k\boldsymbol{R}_2^{\dagger} \end{array}\right), \tag{29}$$

where

$$\boldsymbol{P}_k = \mathrm{diag}\left[q_{\min}, \dots, -\frac{9}{4}\pi, -\frac{\pi}{4}, \frac{7}{4}\pi, \dots, q_{\max}\right],$$

$$\boldsymbol{Q}_k = \mathrm{diag}\left[-q_{\max}, \dots, -\frac{7}{4}\pi, \frac{\pi}{4}, \frac{9}{4}\pi, \dots, -q_{\min}\right],$$

$$q_{\min} = -\frac{\pi}{4} - \left\lfloor \frac{k-1}{2} \right\rfloor 2\pi, \qquad q_{\max} = -\frac{\pi}{4} + \left\lfloor \frac{k}{2} \right\rfloor 2\pi.$$

The counterpart of (19) is

$$H_{2k}' = \left(\begin{array}{c|c} \boldsymbol{R}_k & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{S}_k \end{array}\right), \tag{30}$$

where

$$\boldsymbol{R_k} = \begin{pmatrix} 0 & J_1 & 0 & \dots & 0 \\ J_1 & 0 & J_2 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{k-1} \\ 0 & 0 & 0 & J_{k-1} & (-1)^k J_k \end{pmatrix} \tag{31}$$

$$\boldsymbol{S_k} = \begin{pmatrix} 0 & J_1 & 0 & \dots & 0 \\ J_1 & 0 & J_2 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{k-1} \\ 0 & 0 & 0 & J_{k-1} & (-1)^{k-1} J_k \end{pmatrix}. \tag{32}$$

The eigenvalues of $\boldsymbol{P_k}$ and $\boldsymbol{Q_k}$ in (29) are shifted $+\pi/4$ and $-\pi/4$, respectively relative to the corresponding blocks of H_{pst} of even N, which makes the problem different from the odd N case. Here we only give a calculation method. It can be verified that the eigenvalues of $\boldsymbol{R_k}$ and $\boldsymbol{S_k}$ only differ in a minus sign, which accords with those of $\boldsymbol{P_k}$ and $\boldsymbol{Q_k}$, respectively. Thus, we only need to deal with (31). Since the eigenvalues of $\boldsymbol{R_k}$ are the same as those of $\boldsymbol{P_k}$, we have

$$\text{Det}[\boldsymbol{R_k} - \lambda \boldsymbol{I_k}] = (-1)^k \prod_{i=1}^{N} (\lambda - \lambda_i), \tag{33}$$

$$\text{Tr}[\boldsymbol{R_k}] = (-1)^k J_k = \text{Tr}[\boldsymbol{P_k}], \tag{34}$$

where λ_i's are the eigenvalues of $\boldsymbol{P_k}$ in (29) (i.e. diagonal elements). The left-hand side of (33) can be calculated using a recursion relation:

$$\begin{aligned}\text{Det}[\boldsymbol{R_k} - \lambda \boldsymbol{I_k}] = {} & [-\lambda + (-1)^k J_k]\text{Det}[\boldsymbol{R_{k-1}} - \lambda \boldsymbol{I_{k-1}}]|_{J_{k-1}=0} \\ & - J_{k-1}^2 \,\text{Det}[\boldsymbol{R_{k-2}} - \lambda \boldsymbol{I_{k-2}}]|_{J_{k-2}=0}.\end{aligned}$$

Using (33) and (34), we can calculate J_i's for any even N. Then the general formula, i.e. the second part of equation (7) could be obtained by mathematical induction. For the second solution of even N, the proof is very similar which is omitted here.

5. Conclusion

In this paper, we show that maximal entanglement generation of end qubits in the spin chain of an *XX* model can be realized using a method of pre-engineered inter-qubit couplings. This generation of maximal entanglement can be used to perform quantum cloning between the end qubits. We compare this method with an alternative method based on state transfer through swap operations using quantum gates or through pre-engineered coupling constants for prefect state transfer in [13]. Our method is straightforward and the decrease of fidelity due to imprecise operation time is considerably smaller than a clone-and-swap-scheme. The method used in section IV is an example of inverse eigenvalue problems [30]. Moreover, the state transfer through a spin chain with dissipation has been shown to be possible [31, 32], thus allowing for quantum cloning through a spin chain under realistic dissipation. In the cloning process, the qubits in the middle of a spin chain act as some form of a catalyst, i.e. their state do not change at the end of the cloning process. This role as a catalyst needs further investigation.

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